KESHAV SUTRAVE'S COMPREHENSIVE EXAM

This exam has four parts, each containing two problems. Do one problem from each part.

Problem 1.1 Let M be a manifold, let $\pi : P \to M$ be a principal G-bundle, let V be a finite dimensional vector space, and let $\rho : G \to \mathrm{End}(V)$ be a representation.

1. Explain what is meant by $E = P \times_\rho V$, the vector bundle associated with ρ.
2. What do sections $\psi \in \Gamma(E)$ correspond to in terms of P?
3. Give the definition of a connection on P.
4. Explain how a connection on P induces a covariant derivative ∇ on sections of E.

Problem 1.2 Let (M, g) be a closed, oriented 4-manifold, let $P \to M$ be a principal SU(2)-bundle over M, and let A be a connection on P.

1. Give the definition of F_A, the curvature of A.
2. Explain what it means for A to be anti-self-dual.
3. Prove that anti-self-dual connections on P are absolute minimizers of the Yang-Mills energy
$$\mathrm{YM}(A) = \frac{1}{2}\int_M |F_A|^2 \mathrm{vol}_g.$$
4. Derive the Euler-Lagrange equation of YM and prove by direct computation that anti-self-dual connections satisfy this equation.

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Problem 2.1 Let $n \in \{2, 3, \dots\}$. Consider the n-dimensional sphere

$$S^n := \{x \in \mathbf{R}^{n+1} \ : \ |x| = 1\}$$

equipped with the Riemannian metric g induced by the standard metric on $\mathbf{R}^{n+1}$.

1. Determine a formula for the Levi-Civita connection of (S^n, g).

2. Compute the Riemann curvature tensor of (S^n, g).

3. Determine the geodesics of (S^n, g).

Problem 2.2 Let $n \in \{3, 4, \dots\}$.

1. State Bochner's formula for 1-forms on a Riemannian manifold.

2. Does T^n admit a Riemannian metric with positive Ricci curvature?

Let (M, g) be a Riemannian manifold. Denote by $\cdot^\flat : T^*M \to TM$ the isomorphism induced by the metric. Bochner's formula for Killing fields $v \in \mathrm{Vect}(M)$ reads

$$\nabla^*\nabla v - \mathrm{Ric}_g(v, \cdot)^\flat = 0.$$

3. Suppose that M is closed and $\mathrm{Ric}_g < 0$. Prove that the vector space

   $$\mathfrak{iso}(M, g) := \{v \in \mathrm{Vect}(M) \ : \ \mathscr{L}_v g = 0\}$$

   is trivial.

4. Prove *either* Bochner's formula for harmonic 1-forms or Bochner's formula for Killing fields.

Problem 3.1 Let (M, g) be a closed Riemannian manifold. Denote by

$$\Delta = \mathrm{dd}^* + \mathrm{d}^*\mathrm{d} : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

the Laplace operator on differential forms. Denote by

$$e(t, x, y)$$

the heat kernel of Δ.

1. What sort of object is $e(t, x, y)$? (What kind of section of what bundle?)

2. State two properties of $e(t, x, y)$ that uniquely characterize it.

3. Write down $e(t, x, y)$ as a sum involving the eigenvalues and eigenfunctions of Δ.

Under the heat flow, an L^2 p-form ω on M evolves in time t to a form ω_t.

4. Write a formula for $\omega_t(x)$ as an integral involving $e(t, x, y)$.

5. Denote by $\mathscr{H}$ the space of harmonic forms on (M, g) and denote by $\pi_{\mathscr{H}}$ the L^2 orthogonal projection onto $\mathscr{H}$. Prove that
$$\lim_{t\to\infty} \omega_t = \pi_{\mathscr{H}}(\omega).$$

Problem 3.2 Let

1. Let D be a closed densely defined operator on a Hilbert space H. Give the definition of the adjoint of D.

2. Find the adjoint of $i\frac{\partial}{\partial x}$ on $L^2[0, 1]$, originally defined on $C_0^\infty(0, 1)$. Describe its self-adjoint extensions.

3. What does it mean for an operator to be compact?

4. State the two definitions of Fredholm operators and prove their equivalence.

Problem 4.1

1. State the Riemann-Hurwitz formula.

2. Is there a non-constant holomorphic map $f : \mathbb{C}P^1 \to T^2$? If yes, write down a map; if no, prove why not.

3. Is there a non-constant holomorphic map $f : T^2 \to \mathbb{C}P^1$? If yes, write down a map; if now, prove why not.

4. Prove the Riemann-Hurwitz formula.

Problem 4.2

1. State the uniformization theorem.

2. Let Σ be a closed Riemann surface of genus $g \geq 2$. Prove that the automorphism group
$$\mathrm{Aut}(\Sigma) := \{f : \Sigma \to \Sigma \ : \ f \text{ is biholomorphic}\}$$
is finite.
Hint: You can use problem 2.2.3 and the uniformization theorem.

<u>**Solutions**</u>

Problem 1.1

1. Define the associated vector bundle by
$$E = P \times_\rho V := P \times V/\sim$$
where $(p, v) \sim (pg^{-1}, \rho(g)v)$ for $g \in G$. Since $P/G = M$, and the equivalence relates elements in P which are G-related, this forms a vector bundle over M by the well-defined $\pi_{E\to M}(p, v) := \pi_{P\to M}(p)$.

2. Sections of E can be identified with G-equivariant maps $P \to V$:
$$\Gamma(E) := \{\psi : M \to E \ : \ \pi\psi = \mathrm{id}_M\} = \{\widetilde{\psi} : P \to V \ : \ \widetilde{\psi}(pg^{-1}) = \rho(g)\widetilde{\psi}(p)\}.$$
Understanding this comes from staring at the diagram
$$\begin{array}{ccc} P & \underset{\pi}{\overset{\widetilde{\psi}}{\rightleftarrows}} & P \times V \\ \sim\big\downarrow & & \big\downarrow\sim \\ M & \underset{\pi}{\overset{\psi}{\rightleftarrows}} & E \end{array}$$
and identifying **sections** $P \to P \times V$ with **maps** $P \to V$.

3. There are different definitions of a connection on P. One is a choice of a G-equivariant horizontal space H of TP. That is, for each $p \in P$, let $V_p = \ker \mathrm{d}\pi$ denote the "vertical subspace". Then a connection on P is a choice of $H \subset TP$ such that
   (i) (Horizontal) For each $p \in P$, the following direct sum decomposition holds
$$T_pP = H_p \oplus V_p.$$
   (ii) (Equivariant) For each $p \in P$ and $g \in G$,
$$H_{pg^{-1}} = (R_g)_* H_p$$
   where $R_g(p) = pg^{-1}$ is the G action on P.

4. Note that for $\pi(p) = x \in M$, we have $H_p \cong T_xM$. In other words, each vector $X_x \in T_xM$ has a unique horizontal lift $\widetilde{X}_p \in H_p$, with $(\pi_P)_{*,p}(\widetilde{X}) = X$. Meanwhile, a section ψ of E has a corresponding G-equivariant map $\widetilde{\psi} : P \to V$. So define
$$\nabla_X(\psi) = \mathrm{d}_p\widetilde{\psi}(\widetilde{X}).$$
Here d is a vector version of the exterior derivative d. This is definitely linear in X and using d it satisfies the Liebniz rule on ψ. Since a covariant derivative ∇_X should send $\Gamma(E) \to \Gamma(E)$, we check that the result is G-equivariant.
Let $\pi_P(p) = \pi_P(pg^{-1}) = x \in M$, and let $\widetilde{X}$ and $(R_g)_*\widetilde{X}$ be the horizontal lifts of X at p and pg^{-1} respectively.
$$\nabla_X(\psi)(pg^{-1}) = \mathrm{d}_{pg^{-1}}\widetilde{\psi}\left(\mathrm{d}R_g(\widetilde{X})\right) = \mathrm{d}_p(\widetilde{\psi} \circ R_g)(\widetilde{X}) = \mathrm{d}_p(\rho(g)\widetilde{\psi})(\widetilde{X}) = \rho(g)\mathrm{d}_p(\widetilde{\psi})(\widetilde{X}).$$
Notice d_p passes through the linear map $\rho(g)$.

Problem 1.2

1. Let $\mathfrak{g}$ denote the lie algebra of $SU(2)$, and let ad$P$ denote the adjoint bundle, the vector bundle over M associated to the adjoint representation of G on $\mathfrak{g}$. Then A defines a covariant derivative ∇_A on sections of ad$P$. So $\nabla_A : \Omega^0(\mathrm{ad}P) \to \Omega^1(\mathrm{ad}P)$. One can extend to an exterior derivative d_A (equal to ∇_A on $\Gamma(\mathrm{ad}P) = \Omega^0_M(\mathrm{ad}P)$)
$$\Omega^0_M(\mathrm{ad}P) \xrightarrow{\mathrm{d}_A} \Omega^1_M(\mathrm{ad}P) \xrightarrow{\mathrm{d}_A} \Omega^2_M(\mathrm{ad}P) \xrightarrow{\mathrm{d}_A} \cdots$$
by requiring that
$$\mathrm{d}_A(\omega \wedge \psi) = \mathrm{d}\omega \wedge \psi + (-1)^k \omega \wedge \mathrm{d}_A\psi$$
for $\omega \in \Omega^k_M$ and $\psi \in \Omega^l_M(E)$.
We define $F_A = \mathrm{d}_A^2$ in the following sense: The composition $\mathrm{d}_A \circ \mathrm{d}_A$ turns out to be tensorial. For example, if $\psi \in \Omega^0_M(E)$, then
$$\mathrm{d}_A\mathrm{d}_A(f\psi) = \mathrm{d}_A(\mathrm{d}f \otimes \psi + f\mathrm{d}_A\psi) = -\mathrm{d}f \wedge \mathrm{d}_A\psi + \mathrm{d}f \wedge \mathrm{d}_A\psi + f\mathrm{d}_A\mathrm{d}_A\psi = f\mathrm{d}_A\mathrm{d}_A\psi$$
So there is a (unique) tensor $F_A \in \Omega^2(\mathrm{End}(\mathrm{ad}P))$ such that
$$\mathrm{d}_A\mathrm{d}_A\psi = F_A \wedge \psi$$
where we wedge the forms and also act algebraically.

2. A connection A is called anti-self-dual if it's curvature is an anti-self-dual 2-form, i.e. if $\star$ denotes the Hodge star (note $\star$ extends to vector bundle valued forms $\Omega^k(E) \to \Omega^{n-k}(E)$) then A is ASD if
$$\star F_A = -F_A.$$

3. The 2nd chern class of a vector bundle E can be computed using the curvature of any connection on E
$$-4\pi^2 c_2(E)(M) = -\frac{1}{2}\int_M \mathrm{tr}(F_A \wedge F_A).$$
A matrix $B \in \mathfrak{su}(2)$ is skew-symmetric, so the norm
$$|B|^2 = \mathrm{tr}(B^t B) = -\mathrm{tr}(B^2).$$
In YM we combine this with the norm on forms
$$\langle \omega, \tau \rangle \mathrm{vol}_g = \omega \wedge \star\tau$$
Thus, using that $\star^2 = 1$ for 2-forms
$$-\frac{1}{2}\int_M \mathrm{tr}(F_A \wedge F_A) = -\frac{1}{2}\int_M \mathrm{tr}(F_A \wedge \star(F_A^+ - F_A^-)) = \frac{1}{2}\int_M \left|F_A^-\right|^2 - \left|F_A^+\right|^2 \mathrm{vol}_g$$
where $F_A^\pm$ are the (orthogonal) self-dual and anti-self-dual components of F_A. Now we have
$$\begin{aligned} \left|F_A^-\right|^2 - \left|F_A^+\right|^2 &\leq \left|F_A^-\right|^2 + \left|F_A^+\right|^2 \\ -4\pi^2 c_2(E)(M) &\leq \mathrm{YM}(A) \end{aligned}$$
Since the chern class is topological and thus independent of the choice of connection A, this establishes $-4\pi^2 c_2(E)(M)$ as a lower bound for YM, and we have equality (i.e. the minimum is achieved) for $F_A^+ \equiv 0$ (anti-self-dual connections).

4. Consider family of connections $A + ta$ where A is a connection and $a \in \Omega^1(\mathrm{ad}P)$. Then

$$F_{A+ta} = F_A + t\mathrm{d}_A a + \frac{t^2}{2}[a \wedge a]$$

So the energy

$$\begin{aligned}\mathrm{YM}(A+ta) &= \frac{1}{2}\int_M \langle F_{A+ta}, F_{A+ta}\rangle \mathrm{vol}_g \\ &= \mathrm{YM}(A) + t\int_M \langle F_A, \mathrm{d}_A a\rangle \mathrm{vol}_g + t^2(\cdots)\end{aligned}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{YM}(A+ta) = \langle F_A, \mathrm{d}_A a\rangle_{L^2} = \langle \mathrm{d}_A^* F_A, a\rangle_{L^2}$$

We have a critical point if

$$\mathrm{d}_A^* F_A = 0.$$

Now if F_A is ASD, then

$$\mathrm{d}_A^* F_A = \star \mathrm{d}_A \star F_A = - \star \mathrm{d}_A F_A = 0$$

by the Bianchi identity.

Problem 2.1

1. For a submanifold $S \subset M$ the Levi-Civita connection for the inherited metric looks like projection

$$\nabla^S \varphi = \pi_{M\to S} \nabla^M \varphi.$$

We claim, for $p \in S^n$, $X \in T_p S^n$ and Y a vector field on S^n near p, that

$$\nabla_X Y = \partial_X Y + \langle X, Y\rangle p$$

where the innerproduct is in $\mathbf{R}^{n+1}$, p is thought of as a vector in $\mathbf{R}^{n+1}$, and $\partial_X Y$ is the standard component differentiation, the connection in $\mathbf{R}^{n+1}$. Thus we must prove that $-\langle X, Y\rangle p$ is the normal part of $\partial_X Y$. Since p is a unit vector normal to $T_p S^n$, the normal part is given by

$$\begin{aligned}\langle \partial_X Y, p\rangle p &= (X\langle Y, p\rangle - \langle Y, \partial_X p\rangle)p \\ &= 0 - \langle Y, X\rangle p\end{aligned}$$

where we used that p is the "position vector", i.e. the identity function on $S^n \subset \mathbf{R}^{n+1}$, and Y is always orthogonal to p.

2. We can use this to compute the curvature

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector $X, Y, Z \in T_p S^n$. In this formula we must choose extensions to vector fields, but since R is a tensor it is independent of the choice of extensions, so we choose the constant extensions to $\mathbf{R}^{n+1}$.

Then derivatives in $\mathbf{R}^{n+1}$ vanish everywhere, e.g. $\partial_X Y \equiv 0$. Also remember that $p \perp T_p S^n$.

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla_X(\partial_Y Z + \langle Y, Z\rangle p)\\ &= \partial_X(\langle Y, Z\rangle p) + \langle X, \langle Y, Z\rangle p\rangle\\ &= \partial_X\langle Y, Z\rangle p + \langle Y, Z\rangle X + \langle Y, Z\rangle\langle X, p\rangle\\ &= \langle Y, Z\rangle X.\end{aligned}$$

So

$$\begin{aligned}R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\\ &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z\\ &= \langle Y, Z\rangle X - \langle X, Z\rangle Y\end{aligned}$$

or

$$\langle R(X,Y)Z, W\rangle = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

3. Let $\gamma : I \to S^n \subset \mathbf{R}^{n+1}$ be a geodesic with unit speed. Using part 1 again, the geodesic equation is

$$\begin{aligned}0 &= \nabla_{\gamma'(t)}\gamma'(t)\\ &= \partial_{\gamma'(t)}\gamma'(t) + \langle\gamma'(t), \gamma'(t)\rangle\gamma(t)\\ &= \gamma''(t) + \gamma(t)\end{aligned}$$

The solutions to this (familiar ODE) in $\mathbf{R}^{n+1}$ are

$$\gamma(t) = \gamma(0)\cos(t) + \gamma'(0)\sin(t).$$

The path is a unit circle in the plane spanned by the vectors $\gamma(0), \gamma'(0)$, a great circle (a plane intersected with the sphere).

Problem 2.2

1.

Problem 3.1

1. The heat kernel is a (time-dependent) "double-form". Consider the bundle formed by the two projections $M \times M \to M$:

$$\begin{array}{ccccc} T^*M & & \pi_1^*(T^*M)\otimes\pi_2^*(T^*M) & & T^*M\\ \downarrow & & \downarrow & & \downarrow\\ M & \xleftarrow{\ \pi_1\ } & \mathbf{R}^+ \times M \times M & \xrightarrow{\ \pi_2\ } & M\end{array}$$

The heat kernel e is a section of this bundle. For each $t \in \mathbf{R}^+$ and $x, y \in M$, $e(t,x,y)$ is an element of $T_x^*M \otimes T_y^*M$.

2. The two properties defining the heat kernel:

   (i) $(\partial_t + \Delta_x)e(t,x,y) = 0$

   (ii) $\lim_{t\to 0}\int_M \langle e(t,x,y), \omega(y)\rangle_y \mathrm{dvol}_g(y) = \omega(x)$

4. We will do this part first. The time evolution is

$$\omega_t(x) = e^{-t\Delta}\omega(x) := \int_M \langle e(t,x,y), \omega(y)\rangle_y \ \mathrm{vol}_g(y)$$

3. Let λ_i and ω_i be the eigenvalues and eigenfunctions of Δ, so

$$\Delta\omega_i = \lambda_i\omega_i$$
$$e^{-t\Delta}\omega_i = e^{-t\lambda_i}\omega_i$$

   Let $\omega \in L^2(M)$ and write

$$\omega = \sum a_i\omega_i$$
$$a_i = \langle \omega, \omega_i\rangle_{L^2}.$$

   Evolving through time,

$$\begin{aligned}\omega_t(x) &= e^{-t\Delta}\omega(x)\\ &= \sum_i a_i e^{-t\lambda_i}\omega_i(x)\\ &= \sum_i \left(\int_M \langle \omega_i(y), \omega(y)\rangle \ \mathrm{vol}_g(y)\right) e^{-t\lambda_i}\omega_i(x)\\ &= \int_M \left\langle \left(\sum e^{-t\lambda_i}\omega_i(x)\omega_i(y)\right), \omega(y)\right\rangle_y \ \mathrm{vol}_g(y)\end{aligned}$$

   Thus following part 4,

$$e(t,x,y) = \sum_i e^{-t\lambda_i}\omega_i(x)\omega_i(y)$$

5. Note that each $\lambda_i \geq 0$. Using the eigenfunction decomposition

$$\lim_{t\to\infty}\omega_t = \lim_{t\to\infty}\sum_i a_i e^{-t\lambda_i}\omega_i = \sum_{i,\lambda_i=0} a_i\omega_i.$$

   But $\lambda_i = 0$ is precisely the harmonic condition.

Problem 3.2

1. The domain of the adjoint is defined to be

$$\mathrm{dom}\,(D^*) = \{y \in H \ : \ x \mapsto (Dx, y) \text{ is bounded on } \mathrm{dom}\,(D)\}$$

If $y \in \operatorname{dom}(D^*)$ this means precisely that there is some C_y such that

$$(Dx, y) \leq C_y \|x\|$$

Then it extends to be bounded on $\overline{\operatorname{dom}(D)} = H$.
Another equivalent definition of the domain of D^* is that there is an element, which we call D^*y (necessarily unique because D is densely defined), such that for every $x \in \operatorname{dom}(D)$,

$$(Dx, y) = (x, D^*y).$$

This is the same thing because of the Riesz representation theorem on $\overline{\operatorname{dom}(D)} = H$.

2. The domain of the adjoint is defined to be

$$\begin{aligned} \operatorname{dom}(D^*) &= \{y \in H \ : \ x \mapsto (Dx, y) \text{ is bounded on } \operatorname{dom}(D)\} \\ &= \{y \in H \ : \ \text{There exists } z \in H, \text{ such that } (Dx, y) = (x, z) \text{ for all } x \in \operatorname{dom}(D)\} \end{aligned}$$

These are equivalent by the Riesz representation theorem on $\overline{\operatorname{dom}(D)} = H$. We say $z = D^*y$; such a z is unique by the density of $\operatorname{dom}(D)$.

3. Let's first deal with C_0^∞ functions and figure out what the adjoint should look like. Let $f, g \in C_0^\infty$. Then by integration by parts,

$$(Df, g) = \int i \left(\frac{\partial}{\partial x} f\right) \overline{g} = -\int i f \overline{\left(\frac{\partial}{\partial x} g\right)} = \int f \overline{\left(i \frac{\partial}{\partial x} g\right)} = (f, Dg).$$

In other words, D is a symmetric operator (though this can be guessed from the rest of the question).

4. A bounded operator $K : X \to Y$ is called compact if the image of the unit ball is precompact in Y, that is, for every bounded sequence $\{x_n\} \subset X$, the image sequence $\{Kx_n\}$ has a subsequence which converges to some $y \in Y$ (not necessarily in the image of the ball).

5. Let $T : X \to Y$ be bounded. Then T is Fredholm if either of the following hold.
   (a) The range of T is closed, and $\ker T, \operatorname{coker} T$ are finite-dimensional.
   (b) There is an operator $S : Y \to X$ such that $K_X := \mathrm{Id}_X - ST$ and $K_Y := \mathrm{Id}_Y - TS$ are compact operators on X and Y respectively (i.e. T is invertible up to a compact operator).

(For Hilbert spaces) Assume $\ker T, \operatorname{coker} T \cong R(T)^\perp$ are finite dimensional, (the range is closed as a consequence). Restricting spaces, $T : (\ker T)^\perp \to R(T) = (\operatorname{coker} T)^\perp$ is bijective, and thus has an inverse function T^{-1} (also linear since T is). Moreover, T is closed implies that T^{-1} is closed, and $\operatorname{dom} T^{-1} = R(T)$ is closed by assumption, so the closed graph theorem tells us that T^{-1} is bounded.

Extending by 0, we define the bounded operator $S : Y \to X$ as

$$S := (T^{-1} \oplus 0) : \big(R(T) \oplus \operatorname{coker} T\big) \longrightarrow (\ker T)^\perp \subseteq X.$$

Now K_X and K_Y are the projections to $\ker T$ and $\operatorname{coker} T$ respectively, and thus they are finite rank ($\implies$ compact) operators.

Conversely, suppose T is invertible up to a compact operator. Let us show that $\ker T$ is finite-dimensional by showing that the unit ball $B := B_X(0,1) \cap \ker T$ is compact.

Problem 4.1

1. For a proper nonconstant holomorphic map $f : X \to Y$ of degree d between compact Riemann surfaces,
$$2 - 2g_X = d(2 - 2g_Y) - R_f$$
where R_f is the ramification index: Locally around every point $x \in X$, f can be represented by $z \mapsto z^{d_x}$ for some integer $d_x \geq 1$. Then
$$R_f = \sum_{x \in X} (d_x - 1).$$
(The integer d_x is greater than 1 only for a discrete set of points; this above is really a finite sum over all the critical points of f.)

2. No: Here $g_X = 0, g_Y = 1$, so if such a map existed, then we would have
$$2 - 0 = d(0) - R_f,$$
but $R_f \geq 0$.

3. Yes: What we need is a map that satisfies
$$2 - 2(1) = d(2 - 0) - R_f$$
$$R_f = 2d$$
   1. For a lattice $\Lambda \subset \mathbb{C}$, the Weierstrass $\wp_\Lambda$ function
$$\wp_\Lambda(u) = \frac{1}{u^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{(u - \lambda)^2} - \frac{1}{\lambda^2}$$
is a doubly periodic meromorphic function on $\mathbb{C}$ which descends to a meromorphic function on $T^2 \cong \mathbb{C}/\Lambda$ with a double pole. (Is it injective otherwise, so that $d = 1$?)
   2. By Riemann-Roch, there exists a function on T_2 with (... review condition on zeros/poles ...)
   3. (Algebraic Geometry - also what I understand the least) For an elliptic curve $T^2 \cong X \subset \mathbb{C}P^2$, one can project using a point not on the curve down to $\mathbb{C}P^1 \subset \mathbb{C}P^2$.